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Currents and moduli in the $(4,0)$ theory

Finn Larsen and Emil Martinec

Enrico Fermi Inst. and Dept. of Physics
University of Chicago
5640 S. Ellis Ave., Chicago, IL 60637, USA
E-mail: flarsen@theory.uchicago.edu, ejm@theory.uchicago.edu

Abstract: We consider black strings in five dimensions and their description as a $(4,0)$ CFT. The CFT moduli space is described explicitly, including its subtle global structure. BPS conditions and global symmetries determine the spectrum of charged excitations, leading to an entropy formula for near-extreme black holes in four dimensions with arbitrary charge vector. In the BPS limit, this formula reduces to the quartic $E_{7(7)}$ invariant. The prospects for a description of the $(4,0)$ theory as a solvable CFT are explored.

Keywords: String Duality, Black Holes in String Theory, Supersymmetry and Duality.

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Contents

1. Introduction and summary

The description of black holes as states in certain unitary conformal field theories remains one of the highlights of recent progress in string theory (for reviews see e.g. [1, 2, 3]). The canonical setting for the discussion is the 1+1 dimensional theory obtained from bound states of D1-branes and D5-branes, with the D5 dimensions transverse to the D1 wrapped on a small four-manifold T^4 or $K3$ [4]. This theory has $\mathcal{N} = (4, 4)$ supersymmetry and describes black strings in six dimensions; which compactify to black holes in five dimensions. It is clearly of interest to understand a broader class of theories which are similarly relevant for black holes. The next simplest example is a conformal field theory with $\mathcal{N} = (4, 0)$ supersymmetry that describes black strings in five dimensions, or black holes in four dimensions. This theory is the subject of the present investigation, which may be considered a sequel to our previous work on the D1-D5 system [5]; however, we will see that the $(4, 0)$ theory involves several interesting complications not present in the $(4, 4)$ case.

In the AdS/CFT correspondence [6] the $(4, 4)$ theory is interpreted as string theory on $AdS_3 \times S^3 \times M$ [7, 8, 9], where $M = T^4$ or $M = K3$. The corresponding interpretation of the $(4, 0)$ theory involves string theory on the orbifold $AdS_3 \times S^3/\mathbb{Z}_N \times M$ [10]; or M-theory on $AdS_3 \times S^2 \times X$ [11], where X is some Calabi-Yau three-fold, for which we exclusively take $X = T^6$. The geometries underlying the $(4, 4)$ and $(4, 0)$ theories are therefore quite similar; however, the $(4, 0)$ theory is more challenging. First of all, the supersymmetry is reduced; so many properties of the theory are no longer constrained by general principles. More profoundly, the electric/magnetic duality, special to four dimensions, gives rise to new structures that require special considerations. These interesting structures represent novel elements that have not previously been analyzed in detail.

There are several specific brane configurations that serve as explicit examples of backgrounds, analogous to D1/D5 for the $(4, 4)$ theory. One has three M5-branes intersecting over a common line [12, 13]; another is a type IIB configuration with a fundamental string, a NS5-brane, and a KK-monopole [14, 15]. These backgrounds are introduced in the beginning of section 3 along with more general families, depending on up to 9 background charges. We use all of these examples repeatedly throughout the paper.

The maximally supersymmetric vacuum in five dimensions depends on 42 moduli, parametrizing the coset $E_{6(6)}/\,\mathrm{USp}(8)$ [16]. The backgrounds we consider break some of the supersymmetry, as discussed above, and also restrict the moduli space by fixing the values of some of the moduli in terms of the background charge vector. In section 3.4 we discuss this mechanism in detail, giving the fixed values of the moduli explicitly, and enumerating the moduli that remain undetermined. There are 28 such "free" moduli, parametrizing the coset $F_{4(4)}/\,\mathrm{SU}(2) \times \mathrm{USp}(6)$ [17, 18, 10].

This discussion determines the *local* geometry of the moduli space of the CFTs, but the *global* structure requires more care [19, 5]. In the $(4, 4)$ theory the essential subtlety is captured by a subspace of moduli space which is locally isomorphic to the coset $\mathrm{SL}(2, \mathbb{R})/\,\mathrm{U}(1)$, which can be parametrized by the complex IIB string coupling. In this subspace the global identifications are described by the group $\Gamma_0(N = q_1 q_5)$, acting on the moduli space from the left [5]. These identifications form a genuine subgroup of the $\mathrm{SL}(2, \mathbb{Z})$ identifications under IIB S-duality. In section 3.6, we develop the corresponding description for the $(4, 0)$ theory. We find that the essential structure of moduli space is captured by the five dimensional coset $\mathrm{SL}(3, \mathbb{R})/\,\mathrm{SO}(3)$. Again, the global identifications are a genuine subgroup of the naïve expectation

$SL(3,\mathbb{Z})$. Other features of the moduli space are also described explicitly in terms of the $SL(3,\mathbb{R})/SO(3)$ subspace of moduli space.

A key feature of our discussion is the scaling limit introduced in section 2. In this limit the $(4,0)$ theory decouples from the ambient environment [20]. An important distinction arises between the branes that are interpreted as sources of the background, and those that are charged excitations. The spectrum of the charged excitations is one of our main interests; it is the topic of section 4. We find a total of 27 charges. A part of their spectrum is similar to that of perturbative winding/momentum charges on three independent four-tori; altogether this sector describes 24 charges. The remaining 3 charges are the "electric" duals of the three "magnetically charged" branes that are the sources of the background. These 3 charges are special to the $(4,0)$ theory, and their spectrum is more complicated. We find their conformal weights using a combination of BPS algebra and global symmetries. The results are then used in section 4.3 to compute the entropy of a large class of near-extremal black holes, depending on an arbitrary charge vector in addition to independent mass and angular momentum parameters. The corresponding classical black holes have not yet been constructed in general, and their area is not known. However, our formula for the entropy reduces in the BPS-limit to the quartic invariant of $E_{7(7)}$, as it should. This result provides a fairly intricate test of our microscopic description.

One of the goals of the present investigation is to learn more about the full CFT underlying the system. It is known that the $(4,4)$ theory is described in a region of moduli space by a solvable CFT, namely a sigma-model with a symmetric product orbifold as target space [4]. The $(4,0)$ theory is understood in significantly less detail. For example, it is not known whether there is an underlying solvable CFT and, if so, in what region of moduli space it would be applicable. In the BPS limit many properties are constrained from general principles and a formula has been proposed for the spectrum [21]; however, many features remain mysterious. We find several new facts about the theory which may be helpful in identifying a solvable limit. In section 5 we explore the possibility of a description in terms of a relatively modest variation on the symmetric product idea. Our results are consistent with this working assumption; however, we have not succeeded in determining the precise theory.

The paper is organized as follows. In section 2 we describe the scaling limit defining a theory decoupled from gravity. This leads to the fundamental distinction between branes that are considered part of the background, and those that are excitations. The following sections 3 and 4 describe properties of the background and the excitations, respectively. Each of these sections has a fairly large number of subsections, focussing on specific features. In section 5, we discuss our attempts towards a description of the $(4,0)$ theory as a solvable CFT. Finally, appendix A contains the computation of the basic spectrum of excitations.

2. The scaling limit

We consider four dimensional vacua with maximal supersymmetry; in other words M-theory on T^7. This theory has 56 U(1) charges: 21 from wrapped M5-branes, 21 from wrapped M2-branes, 7 KK-momenta, and 7 KK-monopoles. The $M2/M5$ branes and the KK-momenta/KK-monopoles form dual pairs under electric/magnetic duality in four dimensions.

We are interested in a scaling limit where a theory without gravity decouples form the bulk. The limit is most conveniently introduced on a rectangular torus where it can be defined as [22]:

$$l_p \to 0\,, \qquad \text{with } R_1,\ldots,R_6 \sim l_p\,,\ R_7 \sim 1\,. \tag{2.1}$$

The undemocratic treatment of the toroidal radii introduces a hierarchy among the U(1) charges which is of central importance for our considerations. This is seen by inspecting the masses of isolated objects that carry each of the 56 U(1) charges:[1]

$$M_{M5} = Q_{ij} = \frac{1}{5!}\epsilon_{ijklmno}\frac{R^k R^l R^m R^n R^o}{l_p^6}\,, \tag{2.2}$$

$$M_{M2} = Z^{ij} = \frac{R^i R^j}{l_p^3}\,, \tag{2.3}$$

$$M_{KK} = P^i = \frac{1}{R_i}\,, \tag{2.4}$$

$$M_{KKM} = \tilde{P}_i = \frac{R_i R_1 R_2 \cdots R_7}{l_p^9}\,. \tag{2.5}$$

In the scaling limit the most massive excitations have mass of order l_p^{-3}; they are the KK monopoles with KK direction along the large dimension R_7. The $M2$- and $M5$-branes wrapping R_7, and the KK-monopole not wrapping R_7 have masses of order l_p^{-2}; there are $6+15+6=27$ such excitations. The $M2$-branes and $M5$-branes not wrapping R_7 and the KK momentum along any direction except R_7 have masses of order l_p^{-1}; there are also $6+15+6=27$ of these — as there must be, by electric/magnetic duality. Finally, the KK momentum along R_7 has finite mass.

It is sometimes useful to consider the effective five-dimensional theory which corresponds to avoiding compactification along the "large" dimension R_7 altogether. In this theory the superheavy KK monopoles (with $M \sim l_p^{-3}$) do not exist; this is one of the reasons that we do not consider these objects in this work. The excitations with mass of order l_p^{-2}, l_p^{-1}, and l_p^0 correspond to strings, particles, and waves in the effective five-dimensional theory. They transform under the five-dimensional duality group $E_{6(6)}$ as **27**, **27**, and **1**. The reason that the hierarchy is fundamental for our considerations is that the objects enter very differently in the microscopic description: the excitations with mass of order l_p^{-2}, l_p^{-1}, and l_p^0 are interpreted as background

[1] The precise definitions of the units are $l_p = g^{1/3} l_s$ where $l_s = \sqrt{\alpha'}$.

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fields, charged excitations in the background theory, and neutral chiral excitations. Section 3 is devoted to a discussion of the background, and section 4 considers the remaining U(1) charges, as excitations of the theory governing the background.

The discussion above was for rectangular tori. A more general set of moduli can be restored without further complications, as long as the effective five-dimensional nature of the configuration is respected. This allows for 21 components of a general metric G_{ij} $(i, j, \ldots = 1, \ldots 6)$, 20 components of the three-form field C_{ijk} $(i, j, \ldots = 1, \ldots 6)$, and the pseudo-scalar $\mathcal{E}_{123456}$. ($\mathcal{E}_{IJKLMN}$ is obtained as the potential dual of the 3-form field C_{JKL}.) These 42 scalars parametrize the coset space $E_{6(6)}/\,\mathrm{USp}(8)$, as expected in a maximally supersymmetric vacuum in five dimensions.

3. The background

The background is chosen such that it preserves precisely 1/8 of SUSY. This condition ensures that the effective theory after decoupling is a conformal field theory (CFT) in $1+1$ dimensions, and also that excitations of the background correspond at strong coupling to regular black holes in $3+1$ dimensions. Duality of supergravity ensures that all backgrounds preserving 1/8 SUSY are classically equivalent; in fact, they all correspond to the near-horizon geometry $AdS_3 \times S^2 \times T^6$. That this classical symmetry generalizes to the full quantum description is part of the conjecture embodied in M-theory.

3.1 The canonical backgrounds

In explicit computations it is useful to choose a specific background. One example is three $M5$-branes that intersect over a string, with the string aligned along the "large" dimension (of length R_7) [12, 13] (see the table here on the right).

Brane	1	2	3	4	5	6	7
$M5$	•	•	•	•	–	–	•
$M5$	–	–	•	•	•	•	•
$M5$	•	•	–	–	•	•	•

An advantage of this representation is that it exhibits many symmetries of the situation explicitly. For example, a triality of the three background objects is immediately apparent. This background will be our main example.

Another example is a type IIB string configuration with a KK-monopole, an $NS5$ brane, and a perturbative string $F1$. It is [14, 15] given in the table on the right, where the $\times$ denotes the Taub-NUT direction of the KK-monopole. An advantage of this representation is its close relation to the $D1/D5$ system. (The $F1/NS5$ in the background becomes $D1/D5$ after type IIB S-duality.) Moreover, it is purely NS; so one may consider worldsheet string theory in this background [10]. Note that in the type IIB representation it is R_6 which is "large".

Brane	1	2	3	4	5	6
KKM	•	•	•	•	$\times$	•
$NS5$	•	•	•	•	–	•
$F1$	–	–	–	–	–	•

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The two examples given above are related by the duality chain:

$$M \overset{6-7 \text{ flip}}{\longrightarrow} M \overset{M_{\text{red}} \text{ on } 7}{\longrightarrow} IIA \overset{T_{125}}{\longrightarrow} IIB \overset{S}{\longrightarrow} IIB\,. \tag{3.1}$$

Results obtained in one representation can be translated to the other using this sequence of transformations.

3.2 More general backgrounds

It is often important to consider more general backgrounds. The most general case with all 27 background branes turned on is unfortunately quite complicated. In this section we introduce an intermediate situation where one can turn on 9 different background branes, while still keeping things explicit. The construction will be exploited repeatedly in the sequel.

The idea is that $E_{6(6)}$ has a maximal subgroup $\mathrm{SL}(3)\times\mathrm{SL}(3)\times\mathrm{SL}(3)$, with the **27** decomposing as $(3,3,1)+(1,3,3)+(3,1,3)$. These three sets of background charges can be thought of as matrices:

$$\mathbf{Q_o} = \begin{pmatrix} z_{17} & z_{37} & z_{57} \\ q_{35} & q_{15} & q_{13} \\ \tilde{p}_1 & \tilde{p}_3 & \tilde{p}_5 \end{pmatrix}, \tag{3.2}$$

$$\mathbf{Q_e} = \begin{pmatrix} z_{27} & z_{47} & z_{67} \\ q_{46} & q_{26} & q_{24} \\ \tilde{p}_2 & \tilde{p}_4 & \tilde{p}_6 \end{pmatrix}, \tag{3.3}$$

as well as:

$$\mathbf{Q_5} = \begin{pmatrix} q_{12} & q_{32} & q_{52} \\ q_{14} & q_{34} & q_{54} \\ q_{16} & q_{36} & q_{56} \end{pmatrix}. \tag{3.4}$$

In these formulae the z_{ij} are the integer number of M2-branes wrapping the corresponding cycles; the $\tilde{p}_i$ are KK-monopoles wrapping the compact space, with i the monopole circle; and the q_{ij} are M5-branes with the indices reffering to the dual cycle on the T^7. The E_6 cubic invariant specializes to the sum of the determinants of these three matrices.

The charge matrices (3.2)–(3.4) are related by discrete symmetries. First, the exchange of even and odd subspaces interchanges $\mathbf{Q_o}$ and $\mathbf{Q_e}$; and takes $\mathbf{Q_5} \longrightarrow \mathbf{Q_5^t}$. Next, the sequence of dualities:

$$M \overset{M_{\text{red}} \text{ on } 6}{\longrightarrow} IIA \overset{T_{1345}}{\longrightarrow} IIA \overset{M_{\text{lift}} \text{ on } 6}{\longrightarrow} M\,, \tag{3.5}$$

interchanges $\mathbf{Q_o}$ and $\mathbf{Q_5}$; and takes $\mathbf{Q_e} \to \Gamma \mathbf{Q_e^t} \Gamma$, where:

$$\Gamma = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}. \tag{3.6}$$

Taken together, these tranformations generate a triality map between any two sets of nine charges above.

Under the decomposition $E_{6(6)} \to \mathrm{SL}(3)^3$, the maximal compact subgroup decomposes as $\mathrm{USp}(8) \to \mathrm{SO}(3)^3$. A conveniently described subspace of moduli space is therefore $[\mathrm{SL}(3)/\,\mathrm{SO}(3)]^3$. In the present construction we focus on these 15 moduli, and turn the remaining ones off. This restriction is such that explicit computations remain possible, with results that are representative of the general case with 42 moduli.

Two of the three $\mathrm{SL}(3)/\,\mathrm{SO}(3)$ cosets are the unit volume metrics $\hat{G}_{2i,2j}$ and $\hat{G}_{2i-1,2j-1}$ on the three-subtori of even and odd cycles. These metrics are conveniently written using vielbeins as $\hat{G} = e^t \cdot e$, for which we choose an $\mathrm{SL}(3)/\,\mathrm{SO}(3)$ coset representative of the form $e = A \cdot N$, where A is diagonal with unit determinant and N is upper triangular with unit diagonal. Henceforth we will drop the hats, remembering that the metrics are of unit normalization. The last $\mathrm{SL}(3)/\,\mathrm{SO}(3)$ subspace is parametrized by the volumes V_{135}, V_{246} and constant three-form fields C_{135}, C_{246} on the even and odd subtori, as well as the six-form modulus $\mathcal{E}_{123456}$. The vielbein parametrization is also convenient for describing this part of the moduli space, for which we may write:

$$e_5 = (V_{246}/V_{135})^{1/6} \times \begin{pmatrix} \sqrt{\frac{1}{V_{135}V_{246}}} & & \\ & \sqrt{\frac{V_{135}}{V_{246}}} & \\ & & \sqrt{V_{135}V_{246}} \end{pmatrix} \begin{pmatrix} 1 & C_{135} & \mathcal{E}'_{123456} \\ & 1 & C_{246} \\ & & 1 \end{pmatrix}, \quad (3.7)$$

with the 'metric' $G_5 = e_5^t \cdot e_5$. We introduced $\mathcal{E}'_{123456} = \mathcal{E}_{123456} - \frac{1}{2}C_{135}C_{246}$.

The discrete symmetries also act simply on the moduli. The exchange of odd and even cycles interchanges G_o and G_e; and transforms G_5 as $G_5 \to \Gamma G_5^{-1}\Gamma$. The sequence of dualities (3.5) inverts the odd metric $G_o \to G_o^{-1}$ and interchanges G_e and G_5 according to $G_{e,5} \to \Gamma G_{5,e}\Gamma$.

3.3 The mass of the background

The mass formula for a general background is some $E_{6(6)}$ invariant combination of the charges which is necessarily quite complicated. However, in the various special cases considered in sections (3.1)–(3.2) the details can be carried out explicitly. The most general mass formula follows in principle, by acting on the charges and the moduli with dualities.

We first consider the canonical $M5$-brane background. On a rectangular torus the mass of is simply $M = Q_{12} + Q_{34} + Q_{56}$, the sum of the constituent masses; but in general the mass depends nontrivially on the moduli. In fact, it is a general phenomenon that, in the presence of any configuration of branes, the parity-odd moduli induce additional charges that are not naïvely present. For a general four-dimensional configuration this effect can be taken into account by the shifts [23]:

$$\hat{P}_I = P_I + \frac{1}{2}C_{IJK}Z^{JK} + \left(\frac{1}{4!}C_{JKL}C_{MNI} + \frac{1}{5!}\mathcal{E}_{JKLMNI}\right) Q^{JKLMN},$$

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$$
\begin{aligned}
\hat{Z}^{IJ} &= Z^{IJ} + \frac{1}{3!}C_{KLM}Q^{IJKLM} + \left(\frac{1}{4!}C_{KLM}C_{NPQ} + \frac{1}{5!}\mathcal{E}_{KLMNPQ}\right)\tilde{P}^K \epsilon^{LMNPQIJ}\,, \\
\hat{Q}_{IJ} &= Q_{IJ} + C_{IJK}\tilde{P}^K\,, \\
\hat{\tilde{P}}^I &= \tilde{P}^I\,,
\end{aligned} \tag{3.8}
$$

where $I, J, \ldots = 1, \ldots, 7$, and we use the notation $Q^{IJKLM} \equiv \frac{1}{2}\epsilon^{IJKLMNO}Q_{NO}$. We are interested in the special case where the background is three intersecting $M5$-branes and, as explained in the end of section 2, the moduli are restricted so that the "large" dimension R_7 does not mix with the others. After these specializations the shifts in the charges simplify dramatically, especially because there are no KK-monopoles in the background. The only nontrivial induced charges are:

$$
\hat{Z}^{i7} = \frac{1}{3!}C_{jkl}Q^{ijkl7}\,. \tag{3.9}
$$

The mass of the background $M5$-branes in a vacuum with general moduli can be derived by considering the effect of these induced charges, and further take a general off-diagonal metric into account. After computations similar to those given in appendix A we find:

$$
M^2 = Q_{12}^2 + Q_{34}^2 + Q_{56}^2 + (\hat{\vec{Z}})^2 + 2X\,, \tag{3.10}
$$

where:

$$
\begin{aligned}
X^2 = &\, Q_{56}^2(Q_{12}^2 + (\hat{Z}_{57})^2 + (\hat{Z}_{67})^2) + Q_{12}^2(Q_{34}^2 + (\hat{Z}_{17})^2 + (\hat{Z}_{27})^2) + \\
&+ Q_{56}^2(Q_{34}^2 + (\hat{Z}_{37})^2 + (\hat{Z}_{47})^2) + 2Q_{12}Q_{34}Q_{56}M\,.
\end{aligned} \tag{3.11}
$$

These expressions give a quartic equation for the mass which cannot in general be further simplified. As they stand, (3.10-3.11) presume diagonal metric on the compact torus; however, they can be covariantized to take off-diagonal metrics into account.

We can also construct the mass formula for the more general background discussed in section (3.2). In this case invariants under duality are constructed in matrix form, simply remembering which SL(3)′s act on the left and right. For the background charges, we have for the half-BPS contribution to the mass squared:

$$
\begin{aligned}
M_0^2 = \left[\frac{R_7(V_{135}V_{246})^{2/3}}{l_p^6}\right]^2 &\Big(\mathrm{tr}\left[\mathbf{Q_o}G_o\mathbf{Q_o^t}\Gamma G_5^{-1}\Gamma\right] + \\
&+ tr\left[\mathbf{Q_e}G_e\mathbf{Q_e^t}G_5\right] + \mathrm{tr}\left[\mathbf{Q_5}G_o^{-1}\mathbf{Q_5^t}G_e^{-1}\right]\Big)\,,
\end{aligned} \tag{3.12}
$$

which gives the sum of squares of the background charges with their appropriate dependence on moduli. This expression is covariant under motions in moduli space. Also note that the three terms are permuted by interchange of odd and even cycles, and by the sequence of dualities (3.5).

3.4 Fixed scalars

The background branes are situated in a vacuum described by 42 scalar moduli, parametrizing the coset $E_{6(6)}/\,\mathrm{USp}(8)$. However, in the decoupling limit the value of some of these scalars are determined in terms of the background charges; these are the fixed scalars. The remaining moduli, which remain undetermined, will be referred to as free moduli.

The fixed scalars can be characterized in general using the $N=2$ supersymmetry of the effective $D=5$ theory [24, 25, 17]. The general rule is that the scalars in the hyper-multiplets remain free moduli in the near-horizon theory, while the scalars in the vector-multiplets acquire a mass and become fixed scalars. In the toroidal compactification considered here there are 14 vector-multiplets and 7 hyper-multiplets. The vector-multiplets each have a single scalar and the hyper-multiplets each have four scalars; so this gives 14 fixed scalars and 28 free moduli. The free moduli parametrize the coset $F_{4(4)}/\,\mathrm{SU}(2)\times\mathrm{USp}(6)$ [17, 18, 10].

The physical distinction between fixed scalars and free moduli is exhibited clearly by minimizing the mass of the background over the full moduli space. The scalars determined this way are the fixed scalars; those that remain arbitrary are the free moduli.

The M5 background. To see explicitly how this works, consider the $M5$-brane background. In the simple case of a rectangular torus with vanishing C-fields, the mass is:

$$\begin{aligned} M &= \frac{R_7}{l_p^6}(V_{12}V_{34}q_{56}+V_{12}V_{56}q_{34}+V_{34}V_{56}q_{12}) \qquad (3.13)\\ &= \frac{R_7(V_{12}V_{34}V_{56})^{2/3}}{l_p^6}\left(\left(\frac{V_{12}V_{34}}{V_{56}^2}\right)^{1/3}q_{56}+\left(\frac{V_{12}V_{56}}{V_{34}^2}\right)^{1/3}q_{34}+\left(\frac{V_{34}V_{56}}{V_{12}^2}\right)^{1/3}q_{12}\right), \end{aligned}$$

where $(2\pi)^2V_{ij}$ is the volume of the 2-torus spanning the (ij) cycle and, as before, and q_{ij} is the number of $M5$-branes wrapping the corresponding dual cycle. The coefficient of the second equation is invariant under the dualitites of the effective $D=5$ theory, and so should not be varied. The expression in the large bracket depends only on the two independent ratios V_{12}/V_{34} and V_{34}/V_{56}. Minimizing over these gives:

$$\frac{V_{12}}{V_{34}}=\frac{q_{12}}{q_{34}}\,, \qquad (3.14)$$

$$\frac{V_{34}}{V_{56}}=\frac{q_{34}}{q_{56}}\,. \qquad (3.15)$$

A more symmetric form of the conditions is obtained by noting that the product of the equations gives $V_{12}/V_{56}=q_{12}/q_{56}$. It is simple to verify these values for the fixed scalars by writing the explicit solutions using the harmonic function rule, and taking the near horizon limit.

We now summarize the results for more general moduli. First consider the metric, in the language of complex manifolds. The background $M5$-branes determine a holomorphic structure on the six-torus, pairing the indices (12), (34), and (56). The components of the Kähler metric $G_{\mu\bar{\nu}}$ are the scalars in vector-multiplets, and therefore fixed scalars. An exception is the overall volume $V = \det G_{\mu\bar{\nu}}$ which is a free modulus. (It forms a hypermultiplet together with the free modulus $\mathcal{E}_{123456}$ and two components of the three-form field.) The complex structure $G_{\mu\nu} + \text{h.c.}$ forms 3 hyper-multiplets which give 12 free moduli. Altogether the 21 metric components therefore give 8 fixed scalars and 13 free moduli.

Next, consider the three-form field. Expanding the mass determined through (3.10) for small $\hat{Z}_i$ gives:

$$\delta M = \frac{1}{4Q}\hat{\vec{Z}}^2 \ . \tag{3.16}$$

This shows that the mass has a local minimum when the induced $M2$ brane charge $\hat{Z}_i = 0$. It is clear from (3.9) that there are only 6 linear combinations of C_{ijk} that induce non-trivial $M2$-brane charge; this gives 6 fixed scalar conditions, made explicit as follows. Divide the indices into three sets in accordance with the holomorphic structure, i.e. (12), (34), (56). There are 8 components of C_{ijk} that have one index in each set; all these are free moduli. The remaining 12 components each have two indices within one of the sets, and the remaining index in a different set, e.g. C_{125}. The 6 fields of this kind that are selfdual on the T^4 transverse to the unpaired index are fixed scalars, their anti-selfdual partners are free moduli. For example, $C_{\hat{1}\hat{2}5}+C_{\hat{3}\hat{4}5}$ is a fixed scalar, but $C_{\hat{1}\hat{2}5} - C_{\hat{3}\hat{4}5}$ is a free modulus.

In summary, the 14 fixed scalars are 8 components of the metric and 6 components of the three-form field; and the 28 free moduli are 13 components of the metric, 14 components of the three-form, and the 1 pseudoscalar $\mathcal{E}_{123456}$.

Another way of stating the fixed scalar conditions is that the three physical charges are identical:

$$Q_{12} = Q_{34} = Q_{56} \equiv Q = \frac{R_{11}}{l_P^6} V^{2/3} \left(q_{12}q_{34}q_{56}\right)^{1/3} , \tag{3.17}$$

where the volume $V = V_{12}V_{34}V_{56}$. The mass of the background at the fixed scalar point is $M_{\text{fix}} = 3Q$.

The type IIB background. The distinction between fixed scalars and free moduli is in some ways simpler in the type IIB F1/NS5/KK-monopole background, so we sketch the results in this case too. On a rectangular torus simple minimization, as above, gives two obvious fixed scalars:

$$r_5^2 = \frac{q_5}{q_k} , \tag{3.18}$$

$$\frac{v_4}{g_s^2} = \frac{q_1}{q_5} , \tag{3.19}$$

where r_5 the radius of the KK-monopole direction in string units, v_4 is the volume of the internal T^4 also in string units, and g_s is the string coupling constant. These fixed scalar conditions agree with the result of simply reading off the near-horizon values of the explicit metric, written using the harmonic function rule.

More generally, consider the 26 NS scalars e^{Φ_6} and $G_{ij}+B_{ij}$ (with $i,j=1,\dots 5$). Of these, 20 are free moduli, namely $G_{ij}+B_{ij}$ with $i=1,\dots,4$ and $j=1,\dots,5$; these parametrize the coset $\mathrm{SO}(4,5)/\mathrm{SO}(4)\times\mathrm{SO}(5)$. The remaining 6 NS-moduli are fixed scalars. On a rectangular torus this was shown explicitly for G_{55} and e^{Φ_6} above, and for $G_{i5}-B_{i5}$ in [10].

The 16 RR scalars can be represented as a formal sum of forms $\chi+C^{(2)}+A$. The anti-selfdual part of this form are 8 fixed scalars, and the selfdual part gives 8 free moduli. In components, the fixed scalars are C^-_{ij} and $\chi-A_{1234}$ and $C_{i5}-\frac{1}{3!}\epsilon_{ijkl}A^{jkl}{}_5$ with $i,j\ldots=1,\dots,4$; and the free RR-moduli are given by similar expressions, but with relative signs flipped.

Altogether there are $6+8=14$ fixed scalars and $20+8=28$ free moduli, as there should be.

More general backgrounds. As a final example of the determination of fixed scalars, we consider the more general backgrounds discussed in section (3.2). For a given choice of 9 charges there are two sets of moduli that are relevant; for example, for the pure M5 configurations represented by the matrix $\mathbf{Q_5}$, both the odd and the even metric couple, for a total of 10 moduli; or for the backgrounds involving $\mathbf{Q_o}$, one needs G_o and G_5.

The fixed scalars are determined by extremizing the mass formula for the background. In the present case there are no parity-odd moduli so the induced charges (3.9) vanish. The mass formula becomes:

$$M^2=M_0^2+2X\,, \tag{3.20}$$

where M_0^2 was given in (3.12) and:

$$X^2=\left(\mathrm{tr}[\mathbf{Q_5}G_o^{-1}\mathbf{Q_5^t}G_e^{-1}]\right)^2-\mathrm{tr}[(\mathbf{Q_5}G_o^{-1}\mathbf{Q_5^t}G_e^{-1})^2]+2\mathrm{det}(\mathbf{Q_5})M\,. \tag{3.21}$$

The determinant of $\mathbf{Q_5}$ does not depend on the moduli, and both M_0^2 and X^2 depend only on the combination $Y=\mathbf{Q_5}G_o^{-1}\mathbf{Q_5^t}G_e^{-1}$. The mass M therefore depends only on Y, as far as its dependence on moduli is concerned. Now, one can check that the trace of any power of the quantity $\mathrm{tr}[Y^n]$ is extremized by:

$$G_o=\hat{\mathbf{Q}}_\mathbf{5}^tG_e^{-1}\hat{\mathbf{Q}}_\mathbf{5}\quad,\qquad \hat{\mathbf{Q}}_\mathbf{5}=\frac{\mathbf{Q_5}}{\mathrm{det}[\mathbf{Q_5}]^{1/3}}\,. \tag{3.22}$$

It follows that, whatever the exact expression for the mass in terms of Y, its extremum is (3.22). This shows that, of the two $\mathrm{SL}(3)/\mathrm{SO}(3)$ cosets that are "active" in the 9 charge background, the fixed scalar equations determine one in terms of the other. In the subspace of moduli space considered here there are thus five free moduli, and we are able to find the relation to the other five active moduli explicitly.

3.5 Instability under fragmentation

As discussed in section 3.3, the mass depends nonlinearly on the charges at generic points in moduli space. The background is only unstable under fragmentation when its mass is equal to the sum of its constituent masses so this effect is sufficient to stabilize the configuration. The surface of instability is therefore some lower-dimensional surface in moduli space [26].

The M5-background. To characterize this surface explicitly, consider (3.16) for the mass due to induced charges. In the ground state of the background the fixed scalar conditions tune these charges to zero, e.g. $\hat{Z}^1 = C^{134}Q_{34} + C^{156}Q_{56} = 0$. However, the ratio of charges Q_{34}/Q_{56} is generally different for each of the fragments in the final state, so the decay products are heavier than they would be for vanishing three-form field. This is the effect that stabilizes the background configuration at generic points in moduli space. In order for fragmentation to be allowed, there are 6 conditions of this sort on the three-form field, and 6 additional conditions that arise similarly from the fixed scalar equations for the off-diagonal metric. More precisely there are three hyper-multiplets (with four scalars apiece) that each protect against emission of two kinds of charges, but not the third. If any two of the hyper-multiplets vanish the configuration is unstable. The surface of instability is therefore a codimension 8 subspace of moduli space.

It follows from this reasoning that the background is unstable *everywhere* in moduli space, when all the decay products have the same ratios of charges as the initial state. We avoid this degenerate possibility throughout this work, by assuming that the three background charges are mutually prime.

The type IIB background. It is simpler to characterize the surface of instability in the canonical type IIB configuration F1/NS5/KK-monopole. Here it is manifest that there is a linearly realized $\mathrm{SO}(5,4)$ symmetry acting on the 20 free NS-moduli $G_{ij} + B_{ij}$ (with $i = 1, \ldots, 5,\ j = 1, \ldots, 4$). These moduli do not affect the linear dependence of the mass formula on the charges, so turning on these moduli does not prevent fragmentation. The RR-moduli are quite different: they enter nonlinearly in the way discussed explicitly above for the canonical $M5$-brane configuration. One component of the co-dimension 8 surface of instability is therefore precisely the surface where the 8 free RR-moduli vanish. It is parametrized by the 20 free NS-moduli.

3.6 The global structure of moduli space

The 28 free moduli parametrize a moduli space which is *locally* the non-compact coset $F_{4(4)}/\,\mathrm{SU}(2) \times \mathrm{USp}(6)$. The purpose of this subsection is to discuss the *global* structure of this space. Naively, one might expect to be able to quotient by $F_{4(4)}(\mathbb{Z})$ but, as explained in [19, 5] for the D1-D5 case, this group is too large — it does not leave the set of background masses invariant.

An explicit prescription of the identifications for the D1-D5 system was given in [5]. There, the local structure of the moduli space is $\mathrm{SO}(5,4)/\mathrm{SO}(5)\times\mathrm{SO}(4)$. It is sufficient to consider a representative $\mathrm{SL}(2,\mathbb{Z})\times\mathrm{SL}(2,\mathbb{Z})$ subgroup of the $\mathrm{SO}(5,4,\mathbb{Z})$ duality group. Under this subgroup, a given pair of D1-D5 charges (q_1, q_5) can be mapped to a "canonical background" $(N = q_1q_5, 1)$. The global identifications of the moduli space are those that preserve the charge vector of the canonical background. This subgroup of $\mathrm{SO}(5,4;\mathbb{Z})$ is generated by a particular diagonal $\Gamma_0(N)$ subgroup of $\mathrm{SL}(2,\mathbb{Z})\times\mathrm{SL}(2,\mathbb{Z})$, together with conjugations by elements of the T-duality group. The interesting part of the structure can be projected onto the $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ subspace of the moduli space acted on by one of the $\mathrm{SL}(2,\mathbb{Z})$'s; the fundamental domain projected onto this subspace is then $\Gamma_0(N)\backslash\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$. The moduli space has a cusp for each factorization of N into one-brane and five-brane charges q_1, $q_5 = N/q_1$, where there is a weakly-coupled (large target space) sigma model description of the dynamics. The singular locus, where the D1-D5 bound state can fragment into its constituents, is a codimension four subspace of the moduli space where the sigma model description breaks down (even at large volume) due to singularities in its target space. Under the projection, the singular loci consist of geodesics running between conjugate cusps for D1-D5 charges $(q, N/q)$ and $(N/q, q)$.

One can employ the same strategy in the present context; now the $\mathrm{SL}(3,\mathbb{Z})^3$ subgroup of $E_{6(6)}(\mathbb{Z})$ duality is representative. Many features carry through. First of all, consider a fivebrane charge matrix (3.4):

$$\mathbf{Q_5} = \begin{pmatrix} q_{12} & 0 & 0 \\ 0 & q_{34} & 0 \\ 0 & 0 & q_{56} \end{pmatrix} . \tag{3.23}$$

This can be mapped to a "canonical background" $\mathbf{Q_5} = \mathrm{diag}[N = q_{12}q_{34}q_{56}, 1, 1]$ via the $\mathrm{SL}(3,\mathbb{Z})\times\mathrm{SL}(3,\mathbb{Z})$ transformation $\mathbf{Q_5} \to g_L\mathbf{Q_5}g_R^t$, with:

$$\begin{aligned} g_L &= \begin{pmatrix} a'q_{34}q_{56} & -ab'q_{12}q_{56} & bb'q_{12}q_{34} \\ c' & ad'q_{56} & -bd'q_{34} \\ 0 & c & d \end{pmatrix} , \\ g_R &= \begin{pmatrix} d'q_{34}q_{56} & -dc'q_{12}q_{56} & cc'q_{12}q_{34} \\ b' & da'q_{56} & -ca'q_{34} \\ 0 & b & a \end{pmatrix} , \end{aligned} \tag{3.24}$$

where the coefficients satisfy:

$$adq_{56} + bcq_{34} = a'd'q_{34}q_{56} + b'c'q_{12} = 1 . \tag{3.25}$$

For the existence and uniqueness of this transformation for any (q_{12}, q_{34}, q_{56}) with $q_{12}q_{34}q_{56} = N$, we require that the prime decomposition of N contain any given prime no more than once. Without loss of generality, one can set e.g. $a = b = a' = b' = 1$. The canonical charge matrix is preserved by further transformations generated by a copy of $\Gamma_0(N)$:

$$
\begin{aligned}
\hat{g}_L &= \begin{pmatrix} \alpha & -\beta q_{12}q_{34}q_{56} & 0 \\ \gamma & \delta & 0 \\ 0 & 0 & 1 \end{pmatrix} , \qquad \alpha\delta + \beta\gamma q_{12}q_{34}q_{56} = 1 \\
\hat{g}_R &= \begin{pmatrix} \delta & -\gamma q_{12}q_{34}q_{56} & 0 \\ \beta & \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} ,
\end{aligned}
\tag{3.26}
$$

together with the $\mathrm{SL}(2,\mathbb{Z})$ that acts in the lower right corner. The fixed scalar conditions identify the moduli subspaces acted on by the left and right $\mathrm{SL}(3,\mathbb{Z})$ duality subgroups; the identifications of this five-dimensional space under the action of $\hat{g}$ is representative of the global structure of the moduli space, just as in the D1-D5 case. The full residual duality group $\mathcal{H}_N \subset F_{4(4)}(\mathbb{Z})$ is generated (in the IIB frame where the background charges are purely NS) by the subgroup of $\mathrm{SL}(3,\mathbb{Z})$ analogous to the above, together with T-duality transformations $\mathrm{SO}(5,4;\mathbb{Z})$.

The singular locus in this subspace are the diagonal matrices in $\mathrm{SL}(3,\mathbb{R})/\,\mathrm{SO}(3)$, since then there is no projection of any given fivebrane charge onto the worldvolume of another. From the $A \cdot N$ decomposition, one sees that this is a two-dimensional geodesic submanifold of the five-dimensional homogeneous space. There will be such a submanifold for any unordered triple of fivebrane charges (q_{12}, q_{34}, q_{56}), which gets mapped to a unique geodesic submanifold under the transformation (3.24) to the canonical background. The generic point on the boundary boundary of $\mathrm{SL}(3,\mathbb{R})/\,\mathrm{SO}(3)$ is reached when $A = \mathrm{diag}[\lambda_1, \lambda_2, \lambda_3]$ (with $\lambda_1\lambda_2\lambda_3 = 1$) has one of its eigenvalues degenerate, say $\lambda_1 \to \infty$. Clearly this can happen to any one of the three λ_i, with the possibilities permuted by the action of the Weyl group in $\mathrm{SL}(3)$; this is why the singular loci correspond to unordered triples of constituent charges, since each of these degenerations belongs to the same geodesic submanifold but different orderings of the Q_i.

The singular loci corresponding to different unordered triples (q_{12}, q_{34}, q_{56}) do not intersect, by the same argument given in [5] for the D1-D5 system. Take the background charges to be those of F1/NS5/KK-monopoles in the type IIB description. In this case, the singular locus corresponds to vanishing of all RR moduli, and is left invariant only by T-duality transformations $\mathrm{SO}(5,4;\mathbb{Z})$. Assume that there is a nontrivial intersection, in the canonical background, of the codimension eight singular subspaces corresponding to two different unordered triples of background charges. Pull a point on the intersection back to the original background by the maps specified by one of the two charge sets. The resulting point is parametrized only by NS moduli; but the charges (q_{12}, q_{34}, q_{56}) and $(q'_{12}, q'_{34}, q'_{56})$ cannot be related by T-duality, so we conclude that the singular loci corresponding to different unordered triples of charges are disjoint.

4. The charged excitations

In any background preserving 1/8 of SUSY there is a rich variety of charged excitations, as discussed in section 2. The exploration of this spectrum is one way to learn about the structure of the theory governing the background.

4.1 Introduction to the spectrum of charges

One of the most basic properties of the charged excitations is their spectrum, i.e. the energy as a function of the charge vector. In this section we present heuristic arguments that motivate the spectrum in the case of a rectangular torus; a more precise computation is given in appendix A.

Consider the type IIB F1/NS5/KK-monopole background and include momentum along the fundamental string. The mass of this system is simply the sum of constituent masses:

$$M = Q_K + Q_5 + |\vec{Q}_1 + \vec{P}_{\rm tot}|\,. \tag{4.1}$$

In this formula we have exploited the rotational invariance of the NS5/KK-monopole world-volume to allow a general direction of the vector $\vec{Q}_1 + \vec{P}_{\rm tot}$. In the scaling limit the component of this vector along the "large" dimension R_6 is dominant; it corresponds to the background fundamental string with charge Q_1, and a neutral chiral excitation. We denote the remaining four components $\vec{W}_{F1} + \vec{P}$; these correspond to charged excitations. After expansion, the energy of the charged excitations becomes:

$$E = \frac{1}{2Q_1}(\vec{W}_{F1} + \vec{P})^2\,. \tag{4.2}$$

An important qualitative consequence of this result is that the charged excitations have *finite* energy in the environment created by the background, even though their masses in flat ambient space *diverge* in the scaling limit. It is shown in appendix A that the expression (4.2) remains valid for general orientations of the charge vectors $\vec{W}_{F1}, \vec{P}$.

The duality $ST_{1234}S$ leaves the background invariant, except for the interchange of two charges $Q_1 \leftrightarrow Q_5$. Thus (4.2) implies that the theory has excitations with the spectrum:

$$E = \frac{1}{2Q_5}(\vec{W}_{D1} + \vec{W}_{D3})^2\,, \tag{4.3}$$

where $\vec{W}_{D1}$ is the charge vector of $D1$-branes wrapping 1234; and the $\vec{W}_{D3}$ denote three-branes wrapped within 1234, with the vector index being the direction that is *not* wrapped. The further duality T_{15} similarly leads to the spectrum:

$$E = \frac{1}{2Q_K}(\vec{W}_D + \vec{W}_{\tilde{D}})^2\,, \tag{4.4}$$

where the charge vectors are:

$$\begin{aligned}\vec{W}_D &= (D1_5, D3_{125}, D3_{135}, D3_{145})\,,\\ \vec{W}_{\tilde{D}} &= (D5_{12345}, D3_{345}, D3_{245}, D3_{235})\,.\end{aligned} \tag{4.5}$$

Recall from section 2 that the total number of U(1) charges is 27. The formulae above gives the energy of 24 of these. The remaining three charges are more complicated, because they are the electric-magnetic duals of the charges that appear in the background. The spectrum of these "special" charges:

$$E^{\rm spec} = \frac{1}{2(Q_1+Q_5+Q_K)}(F_5 + P_5 + N_{12345})^2\,, \tag{4.6}$$

is derived in appendix A. It is also shown that the energy of a configuration with general charge vector is the sum of the special cases considered above. The final result for the energy of excitations with general charge vector therefore becomes:

$$\begin{aligned}E = {}& \frac{1}{2Q_1}(\vec{W}_{F1} + \vec{P})^2 + \frac{1}{2Q_5}(\vec{W}_{D1} + \vec{W}_{D3})^2 + \frac{1}{2Q_K}(\vec{W}_D + \vec{\tilde{W}}_D)^2 + \\ &+ \frac{1}{2(Q_1+Q_5+Q_K)}(F_5 + P_5 + N_{12345})^2\,.\end{aligned} \tag{4.7}$$

Microscopic units. So far we have used physical units where the charge is identical to the mass of an isolated brane and is denoted by a capital letter. The microscopic units instead count the number of branes and are denoted by lower-case letters. In these units the energy (4.7) becomes:

$$\begin{aligned}R_6 E = {}& \frac{1}{2q_1}\left(w_i^{F1} r_i + \frac{p^i}{r^i}\right)^2 + \frac{1}{2q_5}\left(w_i^{D1}\frac{r^i}{\sqrt{v_4}} + \vec{w}_{D3}\frac{\sqrt{v_4}}{r^i}\right)^2 + \frac{1}{2q_k}\left(w_i^D e^i + \frac{w^i_{\tilde{D}}}{e^i}\right)^2 + \\ &+ \frac{1}{6q_1q_5q_k}(f_5 q_5 + p^5 q_k + n_{12345} q_1)^2\,,\end{aligned} \tag{4.8}$$

where, in each of the first terms, a sum over the index i is implied *after* taking the square. The last term (containing the special U(1) charges) was rewritten using the fixed scalar equations (3.18)–(3.19). The r^i are radii of the compact dimensions in string units (i.e. $r^i = R_i/l_s$ where $l_s = \sqrt{\alpha'}$), $v_4 \equiv r_1r_2r_3r_4$, and $e^i = \frac{1}{\sqrt{v_4}}(1, r_1r_2, r_1r_3, r_1r_4)$. It is important to note that the scale of the energy is set by the radius of the large dimension R_6.

The M5-background. It is straightforward to find the corresponding formulae for the three M5-branes intersecting over a line, e.g. using the duality (3.1). The result is:

$$\begin{aligned} E = & \frac{1}{2Q_{56}}\left[(Z_{13}+Z_{24})^2+(Z_{14}+Z_{32})^2+(P_5+Q_{67})^2+(P_6+Q_{57})^2\right]+ \\ & +\frac{1}{2Q_{12}}\left[(Z_{35}+Z_{46})^2+(Z_{36}+Z_{54})^2+(P_1+Q_{27})^2+(P_2+Q_{17})^2\right]+ \\ & +\frac{1}{2Q_{34}}\left[(Z_{15}+Z_{26})^2+(Z_{16}+Z_{25})^2+(P_3+Q_{47})^2+(P_4+Q_{37})^2\right]+ \\ & +\frac{1}{2(Q_{12}+Q_{34}+Q_{56})}(Z_{12}+Z_{34}+Z_{56})^2\,, \end{aligned} \tag{4.9}$$

where Z_{ij} are the charges of the M2-branes wrapping the (ij) cycle, P_i is the momentum along R_i, and Q_{ij} denote five-branes transverse to the (ij) cycle. The intersecting $M5$ background makes the systematics of the energy formula clearer: the 3 special U(1) charges are the $M2$-branes that are "electric" duals of the "magnetic" background $M5$'s. The 8 U(1) charges that are weighted by a given background $M5$ (Q_{12}, Q_{34}, or Q_{56}) are the $M2$'s that do not share an index with the background $M5$; and KK-excitations and light $M5$'s that do. These statements translate into simple geometric relations.

4.2 Currents and lattices

The purpose of this section is to take a first step towards an interpretation of the charged spectrum (4.8). The working assumption is that the spectrum arises as the 0-modes of affine currents in the underlying conformal field theory. The AdS/CFT correspondence [6], as implemented for AdS_3 in [27, 28, 29], shows that this is the correct interpretation for the perturbative currents; non-perturbative dualities then suggest that all currents arise in this way.

Consider the first three terms of (4.8). These terms have the structure of 12 U(1) currents in the right-moving sector, 4 currents at each level q_1, q_5, and q_k. Modular invariance of the CFT then requires a matching set of 12 left-moving currents with spectrum:

$$R_6E_L = \frac{1}{2q_1}\left(w_i^{F1}r_i-\frac{p^i}{r^i}\right)^2+\frac{1}{2q_5}\left(w_i^{D1}\frac{r^i}{\sqrt{v_4}}-\vec{w}_{D3}\frac{\sqrt{v_4}}{r^i}\right)^2+\frac{1}{2q_k}\left(w_i^{D}e^i-\frac{w^i_{\tilde{D}}}{e^i}\right)^2. \tag{4.10}$$

Taken together, the right- and left-moving currents form a simple lattice of signature $(12,12)$. The right-moving currents are invariant under the duality T_{1234}; the left-moving ones change sign. It is important to note that these results are reliable only in the semiclassical regime: there is no supersymmetry in the left-moving sector, so the formula (4.10) can not be interpreted as a BPS formula. It can therefore receive corrections; in particular, the levels may receive contributions at the subleading order in the background charge.

Next, we turn attention to the three "special" charges, appearing in the last line of (4.8). An important constraint on these is that the CFT has a global $\mathrm{SU}(2) \times \mathrm{USp}(6)$ symmetry [30]. The full set of charges transforms in the **27** of USp(8) which decomposes as $(\mathbf{2},\mathbf{6})\otimes(\mathbf{1},\mathbf{14})\otimes(\mathbf{1},\mathbf{1})$ under $\mathrm{SU}(2)\times\mathrm{USp}(6)$. The global SU(2) acts on the supersymmetries, so the $(\mathbf{2},\mathbf{6})$ is identified with the 12 right-moving charges. More generally, the energy formula (4.8) is the extremality condition on the right-moving charges. This identifies a specific linear combination of the special charges as right-moving; and we further note that this combination is the singlet $(\mathbf{1},\mathbf{1})$ under $\mathrm{SU}(2)\times\mathrm{USp}(6)$. At this point we have yet to account for the 12 left-moving currents with spectrum given in (4.10), and two linear combinations of the special charges. These must transform as $(\mathbf{1},\mathbf{14})$ under the $\mathrm{SU}(2)\times\mathrm{USp}(6)$ global symmetry and this property determines the spectrum of the special charges as:

$$R_6 E_L^{\mathrm{spec}} = \frac{1}{2q_1q_5q_k}(f_5q_5 - p_5q_k)^2 + \frac{1}{6q_1q_5q_k}(f_5q_5 + p_5q_k - 2n_{12345}q_1)^2 \,. \tag{4.11}$$

The first term in this equation is the dual of the first term in (4.10), after the fixed scalar condition (3.18) is taken into account. This expresses the perturbative $\mathrm{SO}(5,4)$ duality. The second term is determined as the linear combination of charges that are orthogonal both to the perturbative charges in the first term, and the USp(6) singlet; it must be normalized as the other elements in the **14** of USp(6). As a further check on (4.11), note that the total left moving weight is symmetric under triality of the charges, an obvious symmetry in the M-representation.

In addition to the 27 U(1) charges there is the bulk angular momentum of the dual black hole in five dimensions. This charge is interpreted in the CFT as a component of the SU(2) R-charge, with normalization fixed by the supersymmetry algebra [31, 32, 21]. The total spectrum of the "special" right-moving charges therefore becomes:

$$R_6 E_R^{\mathrm{spec}} = \frac{1}{2q_1q_5q_k}J^2 + \frac{1}{6q_1q_5q_k}(f_5q_5 + p_5q_k + n_{12345}q_1)^2 \,. \tag{4.12}$$

There is an obvious similarity between (4.11) and (4.12). This suggests that the special currents are incorporated into the $(12,12)$ lattice found previously so, altogether, the currents form a $(14,14)$ lattice.

There are two peculiar features of the spectrum of excitations (4.12) carrying the special charge: First of all, the normalization differs by a factor of two relative to the other currents (in particular the last term in (4.11)); secondly, although this special charge appears in the right-moving energy, it is associated to the graviphoton tower of supergravity on $AdS_3 \times S^2$, which has the opposite chirality (it is left-moving)[2] [30, 33]. Nevertheless, these results are dictated by BPS algebra and global symmetries.

[2]The latter fact appears to violate the rule of thumb that the singleton sector can be obtained by extending the range of the mode index in the KK tower.

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4.3 Black hole entropy

The excitations that we consider correspond at strong coupling to regular black holes. It is known that the microscopic (statistical) entropy of these black holes agrees with the Bekenstein-Hawking area formula, when only the scalar charge (momentum) is excited. It is interesting to generalize this agreement to the case where the additional U(1) charges are included as well. We carry out the computation in the type IIB background, on a rectangular torus. Some elements of the more general computation are discussed in appendix 4.5, and others are given in [5].

Conformal field theory. For a general background the central charge of the underlying CFT is proportional to the unique cubic invariant of $E_{6(6)}$ that can be formed from the charge vectors in the fundamental representation **27**. For the present purposes it is sufficient to consider the canonical F1/NS5/KK-monopole background; then the central charge is simply $c = 6q_1q_5q_k$.

The operators of the theory have left and right conformal weights $h_{L,R} = \frac{1}{2}(\epsilon \pm p_6)$ where ϵ is the total energy. In a sector with a specified U(1) charge vector the typical vertex operator can be written:

$$V_{\rm tot} = V_{\rm irr}\ V_{\rm U(1)}\,, \tag{4.13}$$

where $V_{\rm U(1)}$ carries the required U(1) charge and $V_{\rm irr}$ is neutral, but otherwise arbitrary. This shows how a part of the conformal weight is expended on exciting the U(1) charges. The unspecified neutral excitations provide the microscopic degeneracy. The corresponding "irreducible" conformal weights are:

$$\begin{aligned} h_L^{\rm irr} &= \frac{\epsilon + p_6}{2} - \frac{1}{4q_1}(\vec{p} - \vec{w}_{F1})^2 - \frac{1}{4q_5}(\vec{w}_{D3} - \vec{w}_{D1})^2 - \frac{1}{4q_k}(\vec{w}_D - \vec{w}_{\tilde{D}})^2 - \\ &\quad - \frac{1}{4q_1q_5q_k}(f_5q_5 - p_5q_k)^2 - \frac{1}{12q_1q_5q_k}(f_5q_5 + p_5q_k - 2n_{12345}q_1)^2 \end{aligned} \tag{4.14}$$

$$\begin{aligned} h_R^{\rm irr} &= \frac{\epsilon - p_6}{2} - \frac{1}{4q_1}(\vec{p} + \vec{w}_{F1})^2 - \frac{1}{4q_5}(\vec{w}_{D3} + \vec{w}_{D1})^2 - \frac{1}{4q_k}(\vec{w}_D + \vec{w}_{\tilde{D}})^2 - \\ &\quad - \frac{1}{12q_1q_5q_k}(f_5q_5 + p_5q_k + n_{12345}q_1)^2 - \frac{1}{4q_1q_5q_k}J^2 \end{aligned} \tag{4.15}$$

where contractions of vectors employ the appropriate moduli-dependent metric, given in section 4.1. The entropy is given in terms of the conformal weights as:

$$S = 2\pi\left[\sqrt{\frac{ch_L^{\rm irr}}{6}} + \sqrt{\frac{ch_R^{\rm irr}}{6}}\right]. \tag{4.16}$$

This is a fairly intricate function of the various black hole parameters.

In the extremal limit the energy ϵ is determined such that $h_R^{\rm irr} = 0$ so the entropy becomes:

$$S = 2\pi\sqrt{q_1q_5q_kh_{\rm ext}^{\rm irr}}\,. \tag{4.17}$$

The left conformal weight can be written:

$$h^{\rm irr}_{ext} = p_6 + \frac{1}{q_1}\vec{p}\cdot\vec{w}_{F1} + \frac{1}{q_5}\vec{w}_{D3}\cdot\vec{w}_{D1} + \frac{1}{q_k}\vec{w}_D\cdot\vec{w}_{\tilde{D}} + \frac{1}{4q_1q_5q_k}J^2 - \tag{4.18}$$
$$- f_5^2\frac{q_5}{4q_kq_1} - p_5^2\frac{q_k}{4q_1q_5} - n_{12345}^2\frac{q_1}{4q_kq_5} + \frac{1}{2q_1}f_5p_5 + \frac{1}{2q_k}f_5n_{12345} + \frac{1}{2q_5}p_5n_{12345}\,.$$

Note that the signs are such that the last terms in this expression do *not* form a complete square.

The area formula. The entropy formula should be compared with the entropy that follows from the area of the corresponding macroscopic black holes. However, the classical solutions needed are difficult to construct, and the task has not been completed. The difficult features are those related to the special charges, i.e. the magnetic duals of the background charges. In the nonrotating case, a generating solution has been constructed which in principle contains the required data [34]. However, it is given in a form which makes the area formula hard to disentangle; for discussion and further references see [35]. In view of these problems we restrict ourselves to the BPS-limit $h^{\rm irr}_R = J = 0$ where the black hole entropy is known on general grounds. It is [36]:

$$S = \pi\sqrt{J_4}\,, \tag{4.19}$$

where J_4 is the quartic invariant of $E_{7(7)}$:

$$-J_4 = x^{ij}y_{jk}x^{kl}y_{li} - \frac{1}{4}x^{ij}y_{ij}x^{kl}y_{kl} + \frac{1}{96}\epsilon_{ijklmnop}(x^{ij}x^{kl}x^{mn}x^{op} + y^{ij}y^{kl}y^{mn}y^{op})\,. \tag{4.20}$$

The invariant is written in the SO(8) formalism where the central charges can be immediately identified with wrapped branes [37]. In M-theory notation the map is $x^{ab} = z^{ab}$, $x^{a8} = \tilde{p}^a$, $y_{ab} = q_{ab}$, $y_{a8} = p_a$ where the indices $a, b, \ldots = 1, \ldots, 7$. After specializing to the background with three $M5$'s intersecting over a line, with further U(1) charges turned on, this becomes:[3]

$$\begin{aligned} J_4 = \;& 4q_{12}q_{34}q_{56}p_7 + \\ & + 4q_{34}q_{56}\left[q_{57}p_6 + q_{67}p_5 + z^{13}z^{24} + z^{14}z^{23}\right] + \\ & + 4q_{34}q_{56}\left[q_{17}p_2 + q_{27}p_1 + z^{35}z^{46} + z^{36}z^{45}\right] + \\ & + 4q_{12}q_{56}\left[q_{37}p_4 + q_{47}p_3 + z^{15}z^{26} + z^{16}z^{25}\right] - \\ & - (q_{12}z^{12})^2 - (q_{34}z^{34})^2 - (q_{56}z^{56})^2 + \\ & + 2q_{12}z^{12}q_{34}z^{34} + 2q_{12}z^{12}q_{56}z^{56} + 2q_{56}z^{56}q_{34}z^{34}\,. \end{aligned} \tag{4.21}$$

The area formula (4.19), with the reduced quartic invariant (4.21), is identical to the microscopic expression (4.17) after dualities. This agreement gives some confidence that the conformal weights have been correctly identified also in the non-BPS case.

[3] The signs are not strictly in accord with (4.20). The convention in this paper is to assign brane (vs. anti-brane) numbers so that a maximal number of terms contribute positively to the entropy; this simplifies dualities.

4.4 Parity-odd moduli

The discussion of the spectrum of charged excitations so far assumed a choice of moduli corresponding to a rectangular torus. It is straightforward to generalize and take into account off-diagonal metric components on the compact space: simply contract indices using the general metric, subject to the fixed scalar conditions.

The parity-odd moduli are more interesting. They induce shifts in the U(1) charges that can be computed using the general rules given in eqs. (3.8). After specializing to the moduli that respect the effective five-dimensional structure these formulae show that the background charges and the charged excitations do not mix. The charged excitations shift via:

$$\begin{aligned}
\hat{P}_i &= P_i + \frac{1}{2} C_{jki} Z^{jk} + \left(\frac{1}{4!} C_{jkl} C_{mni} + \frac{1}{5!} \mathcal{E}_{jklmni} \right) Q^{jklmn} , \\
\hat{Z}^{ij} &= Z^{ij} + \frac{1}{3!} C_{klm} Q^{klmij} , \\
\hat{Q}^{ijklm} &= Q^{ijklm} .
\end{aligned} \tag{4.22}$$

The shifts in the background charges are compensated by changes in the fixed scalars. The net result is therefore that the spectrum of the charged excitations is modified by the parity-odd moduli through the shifts (4.22) of the U(1) charges, but with the background charges unmodified. This rule provides rather detailed information about the structure of the spacetime CFT.

4.5 General moduli and the charged excitations

The moduli dependence of the excitations can be investigated in more detail using the setup introduced in section 3.2. The discussion of the background given there is extended to the excitations by introducing the matrix expressions:

$$\mathbf{Z_o} = \begin{pmatrix} p_1 & p_3 & p_5 \\ z_{35} & z_{15} & z_{13} \\ q_{17} & q_{37} & q_{57} \end{pmatrix} , \tag{4.23}$$

$$\mathbf{Z_e} = \begin{pmatrix} p_2 & p_4 & p_6 \\ z_{46} & z_{26} & z_{24} \\ q_{27} & q_{47} & q_{67} \end{pmatrix} , \tag{4.24}$$

as well as:

$$\mathbf{Z_2} = \begin{pmatrix} z_{12} & z_{32} & z_{52} \\ z_{14} & z_{34} & z_{54} \\ z_{16} & z_{36} & z_{56} \end{pmatrix} . \tag{4.25}$$

It is convenient to discuss the excitations of the type IIB background. Begin with the background consisting of z_{17}, q_{15}, and $\tilde{p}_5$. After M-reduction along R_1 and T-dualization to IIB along R_3, this becomes the canonical F1/NS5/KK-monopole background. We denote the circle dual to the M-theory 13 two-torus by R_B, so that the

T^4 has cycles $B246$. Then we can relabel the corresponding charge matrices as:

$$\mathbf{Z_o} = \begin{pmatrix} D1_B & F1_B & p_5 \\ D1_5 & F1_5 & p_B \\ N5_{B2465} & D5_{B2465} & D3_{246} \end{pmatrix} , \tag{4.26}$$

$$\mathbf{Z_e} = \begin{pmatrix} p_2 & p_4 & p_6 \\ D3_{B46} & D3_{B26} & D3_{B24} \\ D3_{546} & D3_{526} & D3_{524} \end{pmatrix} , \tag{4.27}$$

as well as:

$$\mathbf{Z_2} = \begin{pmatrix} F1_2 & D1_2 & D3_{5B2} \\ F1_4 & D1_4 & D3_{5B4} \\ F1_6 & D1_6 & D3_{5B6} \end{pmatrix} . \tag{4.28}$$

It is straightforward to find the map of the moduli as well.

As we have seen, the structure of the spectrum of excitations is that of winding/momentum charges on a triplet of independent T^4's; in addition, there are three special charges. In the matrices above the $\mathbf{Z_2^t}$ is, entry by entry, the winding dual to the momentum excitation $\mathbf{Z_e}$, for three of the four cycles on the triplet of T^4's; and:

$$\mathbf{Z_o}\Gamma = \begin{pmatrix} p_5 & F1_B & D1_B \\ p_B & F1_5 & D1_5 \\ D3_{246} & D5_{B2465} & N5_{B2465} \end{pmatrix} , \tag{4.29}$$

has the remaining three momentum-winding pairs (the pairing is under reflection across the diagonal), as well as the three special charges on the diagonal. The triality of the three T^4's acts by:

$$\mathbf{Z_e} \to \Omega \mathbf{Z_e} \quad , \qquad \mathbf{Z_2} \to \mathbf{Z_2}\Omega^{-1} \quad , \qquad \mathbf{Z_o}\Gamma \to \Omega \mathbf{Z_o}\Gamma\Omega^{-1} , \tag{4.30}$$

where Ω is a permutation matrix. The fact that this operation permutes the special charges as well suggests that each of the three special charges should be associated with a particular T^4.

The spectrum. The half-BPS contribution to the masses of the excitations is:

$$R_6 E_0 = \frac{1}{2}\left(\mathrm{tr}\left[\mathbf{Z_o} G_o^{-1} \mathbf{Z_o^t} G_5\right] + \mathrm{tr}\left[\mathbf{Z_e} G_e^{-1} \mathbf{Z_e^t} \Gamma G_5^{-1} \Gamma\right] + \mathrm{tr}\left[\mathbf{Z_2} G_o \mathbf{Z_2^t} G_e\right]\right) . \tag{4.31}$$

Specializing this expression to rectangular tori and exploiting the fixed scalar conditions, we recover appropriate denominators in the mass formula. For generic moduli there is not a meaningful concept of "levels", i.e. integer denominaters in the mass formula. However, it follows from the discussion in section 3.6 that other specializations of the moduli, corresponding to $\mathrm{SL}(3,\mathbb{Z})$ transforms of the rectangular tori, likewise give simple denominators. That this also works out in the present formulae is a consequence of covariance under duality transformations.

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The cross-terms in the mass formula come from 1/4-BPS and higher contributions and are independent of the moduli. They are determined as the expressions involving charges that transform in compatible ways and reduce appropriately in the special case of rectangular tori. The result:

$$\begin{aligned} R_6 E_\times &= \mathrm{tr}\left[\mathbf{Z_2 Q_o^{-1} Z_e}\right] + \mathbf{Z_o}^{ia}\mathbf{Z_o}^{jb}\left(\Gamma(\mathbf{Q_o^t})^{-1}\right)^{ck}\epsilon_{ijk}\epsilon_{abc} + \\ &\quad + \text{triality permutations of } (\mathrm{o}, \mathrm{e}, 2/5)\,, \end{aligned} \tag{4.32}$$

is invariant, independent of moduli, and has the appropriate denominators in the expected places. Finally, the quantity:

$$R_6 E_{\mathrm{spec}} = \frac{\left(\mathrm{tr}\left[\mathbf{Q_o^t}\Gamma\mathbf{Z_o} + \mathbf{Q_e^t}\Gamma\mathbf{Z_e} + \mathbf{Q_5^t Z_2}\right]\right)^2}{6\ \left(\det\mathbf{Q_o} + \det\mathbf{Q_e} + \det\mathbf{Q_5}\right)}\,, \tag{4.33}$$

is the square of the sum of special charges, i.e. $(\tilde{p}_5 p_5 + n_{B2467} f_5 + f_7 n_{B2465})$. Thus we can write, for example:

$$\begin{aligned} E_L &= E_0 - 2E_\times + E_{\mathrm{spec}}\,, \\ E_R &= E_0 + 2E_\times - 2E_{\mathrm{spec}}\,. \end{aligned} \tag{4.34}$$

5. Is there an exactly solvable CFT in the moduli space?

In the AdS/CFT correspondence, it is extremely useful to find a point in the moduli space where the CFT is amenable to perturbative treatment, or even better, exactly solvable. For instance, invariants on the moduli space, such as the BPS spectrum and its degeneracy, quantum corrections to current algebra anomalies, etc., can be worked out explicitly. Thus one may ask whether the three-charge brane background we have been discussing admits an exactly solvable CFT in the moduli space.

Let us recall the structure of the symmetric product orbifold CFT that appears in the moduli space of the two-charge (e.g. D1-D5) brane background, given its remarkable similarity to the present problem. For T^4, the CFT in question is a sigma model on $Sym^N(T^4)\times(\tilde{T}^4\times\tilde{\mathbb{R}}\times\tilde{S}^3)$, with $N = q_1 q_5$; it describes a region in the moduli space that is naturally associated to background brane charges $(q_1, q_5) = (N, 1)$. The other factors in the target space represent the zero modes of the diagonal U(1) in the $\mathrm{U}(q_1)\times\mathrm{U}(q_5)$ gauge theory on the D1- and D5-branes; the $\tilde{\mathbb{R}}\times\tilde{S}^3$ part, representing motion transverse to the T^4, decouples and may be ignored in this instance. The CFT moduli space is $\mathcal{H}_N\backslash\,\mathrm{SO}(5,4)/\,\mathrm{SO}(5)\times\mathrm{SO}(4)$, with $\mathcal{H}_N$ the discrete subgroup of $\mathrm{SO}(5,4;\mathbb{Z})$ that preserves the background charge vector [19, 5]. BPS considerations [5] and an analysis of the gravitational effective field theory [29] show the existence of a $\mathrm{U}(1)^8_L\times\mathrm{U}(1)^8_R$ current algebra; half of these come from the diagonal of the symmetric orbifold, half from the extra $\tilde{T}^4$. These currents are a part of the singleton sector of the CFT. Moduli deformations affect the CFT in two ways: First, they act as deformations of the T^4 components of the symmetric product orbifold (sixteen of the moduli are the metric and B-field of the individual T^4's, and four are $\mathbb{Z}_2$ twist operators); roughly speaking, the various twisted sectors realize supergravity in the bulk of $AdS_3 \times S^3$. Secondly, the moduli act in the global, singleton sector via current-current interactions which deform the metric on the $\mathrm{U}(1)^{16}$ charge lattice. The spectrum of U(1) charged excitations in the symmetric orbifold, as well as other considerations, is sufficient to identify the symmetric orbifold locus in a corner of the moduli space that corresponds to canonical background charges $(N, 1)$ [5]. Deformations away from this locus are a linear combination of the moduli of the symmetric product, and the current-current interactions.

The addition of KK monopoles to the background leads to a remarkably parallel structure to the moduli space, as we have seen. It is therefore tempting to speculate that there is again an exactly solvable point in the moduli space in the region corresponding to the canonical background charges $(N, 1, 1)$. The near-horizon geometry of one KK monopole is the same as flat space; in particular, it does not deform the angular S^3 of the space transverse to the onebrane-fivebrane system. This leads one to suppose that the symmetric orbifold structure survives more or less intact.

The moduli space is now $\mathcal{H}_N \backslash F_{4(4)} / \mathrm{SU}(2) \times \mathrm{USp}(6)$. In addition to the twenty moduli of the D1-D5 system, there are eight additional moduli; four of these come from the enlargement of the T-duality group from $\mathrm{SO}(4,4)$ to $\mathrm{SO}(5,4)$ (mixing the fibered circle of the KK monopole with the four-torus), and four more are RR moduli.[4] The spacetime CFT has twelve right-moving currents in the $(\mathbf{2},\mathbf{6})$ of $\mathrm{SU}(2) \times \mathrm{USp}(6)$, and fourteen left-moving currents in the $(\mathbf{1}, \mathbf{14})$. In the IIB description, these currents naturally fall into three sets of $\mathrm{U}(1)^4_L \times \mathrm{U}(1)^4_R$, together with two additional 'special currents' on the left. One of these sets of $\mathrm{U}(1)^4_L \times \mathrm{U}(1)^4_R$ corresponds to perturbative string momentum and winding (4.2), and is naturally associated with the one-brane background; another set (4.3) consists of D1- and D3-branes, and is naturally associated with the fivebrane background [29, 5] and the third set (4.4) is associated to the KK monopole background. We propose to again identify the first set of currents with the translation currents on the diagonal T^4 of the symmetric product $Sym^N(T^4)$, while the other two sets are tentatively identified with two extra four-tori, denoted $\tilde{T}^4$ and $\hat{T}^4$ (two (4,0) hypermultiplets are the minimal additional field content needed to realize these currents).

Fivebrane anomalies [38, 39] have terms linear and cubic in the brane charges. Since there is no constant term, the contribution to the anomaly in the SU(2) R-current coming from the extra $\tilde{T}^4 \times \hat{T}^4$ hypermultiplets must be cancelled; the simplest possibility is to add vectormultiplets, which might be thought of again as representing zero-modes of the bound state in the $\mathbb{R} \times S^3$ throat transverse to the four-torus.

[4]When the background charges are entirely NS.

One important problem is to realize the structure of the moduli space in this framework. The 28 moduli of $F_{4(4)}/\,\mathrm{SU}(2)\times\mathrm{USp}(6)$ transform as a $(\mathbf{2},\mathbf{14'})$ under $\mathrm{SU}(2)\times\mathrm{USp}(6)$. They again appear in two ways: As moduli of the symmetric product orbifold, and as current-current interactions. The T-duality group $\mathrm{SO}(5,4)$ is manifestly realized in the symmetric orbifold, when we supplement the $\mathrm{U}(1)_L^4\times\mathrm{U}(1)_R^4$ translation currents on each individual four-torus with the J_3 component of the left-moving SU(2) R-symmetry.[5] The four $\mathbb{Z}_2$ twist moduli that preserve the $\mathcal{N}=(4,4)$ supersymmetry of the symmetric product may be supplemented with four more such moduli that only preserve $\mathcal{N}=(4,0)$ supersymmetry. These moduli were identified in [5] by matching quantum numbers in the symmetric orbifold to those of the supergravity spectrum. One also needs to represent the effect of the moduli on the charged spectrum of the singleton sector. The marginal deformations of a set of U(1) currents are always of the form $\mathrm{SO}(p,q)/\,\mathrm{SO}(p)\times\mathrm{SO}(q)$; therefore, the $F_{4(4)}/\,\mathrm{SU}(2)\times\mathrm{USp}(6)$ local geometry of the moduli space must, as far as the singleton sector is concerned, embed in such a structure. Indeed, $F_{4(4)}$ embeds in $\mathrm{SO}(12,14)$ via the $\mathbf{26}$, while $\mathrm{SU}(2)\times\mathrm{USp}(6)$ embeds in SO(12) via the $(\mathbf{2},\mathbf{6})$ and in SO(14) via the $(\mathbf{1},\mathbf{14})$. Note that $(\mathbf{2},\mathbf{6})\otimes(\mathbf{1},\mathbf{14})\supset(\mathbf{2},\mathbf{14'})$; there is a unique projection of the current bilinears onto the appropriate subspace of $\mathrm{SO}(12,14)$.

Clearly $T^4_{\mathrm{diag}}\times\tilde{T}^4\times\hat{T}^4$ realizes an $\mathrm{SO}(12,12)$ structure; but one must identify two more left-moving currents to fill out the $\mathbf{14}$ of USp(6). From (4.10), one current is (momentum/winding) of perturbative strings on the KK monopole circle. The $\mathrm{SO}(5,4)$ T-duality group mixes this current with the analogous ones on T^4. Under T-duality, one has the decomposition:

$$\begin{array}{ccccc} F_{4(4)} & \to & \mathrm{SO}(5,4) & \to & \mathrm{SO}(5)\times\mathrm{SO}(4) \\ \mathbf{26} & & \mathbf{9}+\mathbf{16}+\mathbf{1} & & [(\mathbf{5};\mathbf{1},\mathbf{1})+(\mathbf{1};\mathbf{2},\mathbf{2})]+ \\ & & & & +[(\mathbf{4};\mathbf{1},\mathbf{2})+(\mathbf{4};\mathbf{2},\mathbf{1})]+(\mathbf{1};\mathbf{1},\mathbf{1})\ . \end{array}$$

The natural realization of $\mathrm{SO}(5,4)$ on the global modes of the candidate CFT mixes the diagonal T^4 translation currents with the overall J_L^3 of the symmetric product; this accounts for the first line in the last column. The second pair of representations in brackets consists of the translation currents of $\tilde{T}^4\times\hat{T}^4$; the remaining singlet must be made from some left-moving current, since it is the last element of the $\mathbf{14}$. Note that the T-duality $\mathrm{SO}(5,4)$ naturally embeds in the $\mathrm{SO}(8,8)$ T-duality group of $\tilde{T}^4\times\hat{T}^4$ via the spinor representation (i.e. the sixteen translation currents transform as the vector of $\mathrm{SO}(8,8)$ and the spinor of $\mathrm{SO}(5,4)$). This is precisely the way in which D-brane charges transform under T-duality. The $\mathrm{SO}(5,4)$ structure is thus manifestly realized in the candidate CFT.

[5] Indeed, it is precisely this quantum number that is affected by the asymmetric orbifold which introduces KK monopole charge in pertubative string descriptions [40, 15, 10].

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Modular invariance suggests completion of the signature (12,14) lattice transforming under $F_{4(4)}$, to a (14,14) lattice as explained in section 4.2; the extra charges are the special BPS charge (4.4), and the $SU(2)_R$ R-charge. The latter is natural, since it contains the right-moving partner of J_L^3, the current of translations on the KK monopole circle. A puzzling feature of the spectrum derived in section 4.2 is the difference in left- and right-moving levels, which is an invariant on the moduli space. The contribution of the charges to this difference is:

$$\begin{aligned} Q_R^2 - Q_L^2 = & \frac{1}{q_1}(p^i w_i) + \frac{1}{q_5}(w_i^{D1} w_{D3}^i) + \frac{1}{q_k}(w_i^D w_{\tilde{D}}^i) + \\ & + \frac{1}{4q_1 q_5 q_k}\Big[J^2 - (q_k p_5)^2 - (q_5 w^5)^2 - (q_1 n_{12345})^2 + \\ & \quad + 2(q_5 q_k) p_5 w^5 + 2(q_k q_1) p_5 n_{12345} + 2(q_1 q_5) w^5 n^{12345}\Big] . \end{aligned} \tag{5.1}$$

Matching this structure places strong constraints on the spacetime CFT, and in particular the currents that remain to be identified in the proposed candidate — the current that couples to the special BPS charge on the right, and the last member of the **14** on the left.

In the $(4,4)$ theory — the D1/D5-system — the BPS states of the symmetric product orbifold match those of the KK towers of the effective 6d supergravity on $AdS_3 \times S^3$ [41, 30, 33]. The SU(2) R-symmetry current algebra quantum numbers label the spherical harmonics on S^3. The 5d supergravity on $AdS_3 \times S^2$ that results when KK monopoles are added to the background is simply the truncation of this spectrum to $J_L^3{=}0$, in the sector with vanishing momentum along the KK fiber circle. Deformation in the moduli space away from the $(N,1,1)$ corner will 'squash' the S^3, so that by the time one is in a region with a valid low-energy supergravity interpretation, the states carrying momentum J_L^3 on the monopole circle will be much heavier that those with $J_L^3 = 0$. Note that the squashing of the three-sphere will act as well on the two vectormultiplets describing the $\mathbb{R}\times S^3$ throat transverse to the four-torus parametrized by the hypermultiplets, suggesting that the last element of the left-moving **14** should also involve the bosonic SU(2) WZW currents.

There are other requirements on the spacetime CFT. One can analyze the F1-NS5-KK monopole background in global coordinates on AdS_3 [10], using the perturbative string techniques of [27].[6] At least a subset of the BPS spectrum should be realized in this framework [19]. In [10], BPS states carrying the quantum numbers of perturbative strings with winding and momentum on the KK monopole circle (parametrized by x^5) were identified; these have dimension $h_R = j_R = j_R^3$ and carry oscillator excitation level $N_{\rm osc} = p_5 w^5$, and thus their degeneracy is exponential in $\sqrt{p_5 w^5}$.

[6]Note that this approach describes the background only on singular loci, and for $q_k > 1$, a rather different subspace of the moduli space; hence one can only safely compare invariant quantities such as BPS spectra.

Thus far, we have not been able to find a suitable modification of the candidate $Sym^N(T^4) \times (\tilde{T}^4 \times \tilde{\mathbb{R}} \times \tilde{S}^3) \times (\hat{T}^4 \times \hat{\mathbb{R}} \times \hat{S}^3)$ CFT meeting all the above requirements. The crucial missing ingredient is an identification of the proper linear combination of the many available currents which realizes the structure of the 'special' charges p_5, w^5 and n_{12345}.

Comments on K3. Finally, we should remark on the differences between compactification on T^4 and K3. For D1-D5 bound states on K3, there was rather little difference; the extra $\tilde{T}^4$ was simply replaced by an extra factor in the symmetric product, and the duality group was still sufficient to place all brane charges (q_1, q_5) with the same product $N = q_1 q_5$ within the same moduli space. The addition of KK monopoles to the mix yields a difference; the duality group is no longer sufficient to map all brane charges to $(N, 1, 1)$ — there is no duality that mixes the KK monopole charge with the other two, as there is in a T^4 compactification. This is reflected in the presence of a linear term $q_k(q_1 q_5 + 2)$ in the anomaly of the R-current. A consequence is that one expects to have the possibility of an exactly solvable point in the moduli space only for $q_k = 1$.

The 5d supergravity on $AdS_3 \times S^2 \times (S^1 \times K3)$ inherits 22 vectors from tensor multiplets of IIB on K3, two from each of the two gravitino multiplets, and one additional vector (the graviphoton) from the supergravity multiplet [42]. The moduli space is locally $\mathrm{SO}(21,4)/\,\mathrm{SO}(21) \times \mathrm{SO}(4)$, with the charges transforming as $(\mathbf{1}, \mathbf{21}) \oplus (\mathbf{2}, \mathbf{2}) \oplus 2(\mathbf{1}, \mathbf{1})$ under $\mathrm{SO}(4) \times \mathrm{SO}(21)$. The first two consist of the (4,20) lattice of D1/D3/D5-branes wrapping the monopole circle and cycles of K3, together with one linear combination of the special charges; one singlet is the other combination of special charges coupling to a left-moving current; and the remaining singlet is the special BPS charge dual to the brane background, which couples to a right-moving current. Adding the R-current, the lattice has signature $(6, 22)$. The four multiplets of charges each have components in common with the toroidal case; so global symmetries, and the discussion of the latter case in this paper, combine to ensure that all the charges are associated to currents.

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A. Derivation of a BPS mass formula

The computation in this appendix follows [23]; see also the appendix of [5].

The supersymmetry algebra in M-theory leads to the eigenvalue equation for the central charges:

$$\left[\mathcal{C}\Gamma^M P_M + \frac{1}{2}\mathcal{C}\Gamma_{MN}Z^{MN} + \frac{1}{5!}\mathcal{C}\Gamma_{MNPQR}Q^{MNPQR}\right]\epsilon = 0\,. \tag{A.1}$$

The central charges Z_{MN} and $Q^{MNPQR} \equiv \frac{1}{2}\epsilon^{MNPQRST}Q_{ST}$ are the M2- and M5-brane charges; the P_M are the momenta, in particular P_0 is the mass M that we want to compute. The spinorial eigenvector of the preserved supersymmetry is denoted ϵ, and the metric is mostly plus.

We choose the background as the three $M5$-branes intersecting over a line, and want to consider any configuration of U(1) charges. Writing out the eigenvalue equation in terms of individual charges yields an expression that is lengthy and not illuminating. Motivated by the qualitative considerations in the main text, the charges can be divided into various groups (see table 1)

We consider first the terms that correspond to the third line in the table, that is a single background $M5$ and 8 specific U(1) charges. The eigenvalue equation becomes:

$$\begin{aligned}\lambda_3 = Q_{56}\Gamma^{012347} &+ \left[\Gamma^{05}P_5 + \Gamma^{06}P_6 + \Gamma^{012346}Q_{57} + \Gamma^{012345}Q_{67}\right] + \\ &+ \left[\Gamma^{013}Z_{13} + \Gamma^{024}Z_{24} + \Gamma^{013}Z_{14} + \Gamma^{013}Z_{23}\right]\,,\end{aligned} \tag{A.2}$$

which is understood as an operator equation acting on the spinor ϵ. The three terms on the right hand side are mutually anti-commuting, so the square of the equation becomes:

$$\begin{aligned}\lambda_3^2 &= Q_{56}^2 + \left[\Gamma^{05}P_5 + \Gamma^{06}P_6 + \Gamma^{012346}Q_{57} + \Gamma^{012345}Q_{67}\right]^2 + \\ &\quad + \left[\Gamma^{013}Z_{13} + \Gamma^{024}Z_{24} + \Gamma^{013}Z_{14} + \Gamma^{013}Z_{23}\right]^2 \\ &= Q_{56}^2 + (P_5 + Q_{67})^2 + (P_6 + Q_{57})^2 + (Z_{13} + Z_{24})^2 + (Z_{14} + Z_{32})^2\,.\end{aligned} \tag{A.3}$$

In the computation leading to the second expression we took $\Gamma^{1234} = 1$, thus removing all operators. We can therefore take an ordinary square root, and then expand the result according to the hierarchy between the charges. This gives:

$$\lambda_3 \simeq Q_{56} + \frac{1}{2Q_{56}}\left[(P_5 + Z_{67})^2 + (P_6 + Z_{57})^2 + (Z_{13} + Z_{24})^2 + (Z_{14} + Z_{32})^2\right]\,. \tag{A.4}$$

level	$M2$	KK	$M5$
Q_{12}	$Z_{35}, Z_{46}, Z_{36}, Z_{45}$	P_1, P_2	Q_{17}, Q_{27}
Q_{34}	$Z_{15}, Z_{26}, Z_{16}, Z_{25}$	P_3, P_4	Q_{37}, Q_{47}
Q_{56}	$Z_{13}, Z_{24}, Z_{14}, Z_{23}$	P_5, P_6	Q_{57}, Q_{67}
$Q_{12} + Q_{34} + Q_{56}$	Z_{12}, Z_{34}, Z_{56}		

Table 1: ??? which caption ???

An analogous computation can be carried out for the charges in the first and second line of the table, yielding eigenvalues λ_1 and λ_2, respectively, after imposing the conditions $\Gamma^{3456} = 1$ and $-\Gamma^{1256} = \Gamma^{1234}\Gamma^{3456} = 1$. Now note that the terms arising from any of the first three lines of the table commute with those coming from the others, in particular Γ^{1234}, Γ^{3456}, and Γ^{1256} mutually commute. Therefore the involved matrices are simultaneously diagonalisable or, in other words, the full result is simply the sum of three terms of the form (A.4).

At this point we need to include also the three "special" U(1) charges that appear in the fourth line of the table. Let us consider these together with the background, without other U(1) charges present. The eigenvalue problem becomes:[7]

$$\mu = Q_{12}\Gamma^{034567} - Q_{34}\Gamma^{012567} + Q_{56}\Gamma^{012347} + Z_{12}\Gamma^{012} - Z_{34}\Gamma^{034} + Z_{56}\Gamma^{056} . \qquad \text{(A.5)}$$

The first three and the last three terms commute between themselves, but these two groups anticommute with each other. The square therefore gives:

$$\mu^2 = (Q_{12} + Q_{34} + Q_{56})^2 + (Z_{12} + Z_{34} + Z_{56})^2 , \qquad \text{(A.6)}$$

for $\Gamma^{1234} = \Gamma^{3456} = -\Gamma^{1256} = 1$. This is an exact expression for the mass, when only the background and the three special U(1) charges are turned on. In the scaling limit it becomes:

$$\mu \simeq Q_{12} + Q_{34} + Q_{56} + \frac{(Z_{12} + Z_{34} + Z_{56})^2}{2(Q_{56} + Q_{12} + Q_{34})} . \qquad \text{(A.7)}$$

The transformations that diagonalize the eigenvalue problems for $\lambda_{1,2,3}$ commute with each other, as remarked above, but they do not in general have a simple relation to the eigenvalue problem for μ. However, to the order we compute the eigenvectors of the undisturbed background solve all the eigenvalue problems considered; and this is sufficient to guarantee that the various partial results can be added without inducing further crossterms. The final result for the mass is therefore:

$$\begin{aligned} M \simeq\ & Q_{12} + Q_{34} + Q_{56} + \\ & + \frac{1}{2Q_{12}} \left[(P_1 + Q_{27})^2 + (P_2 + Q_{17})^2 + (Z_{35} + Z_{46})^2 + (Z_{36} + Z_{5E4})^2 \right] + \\ & + \frac{1}{2Q_{34}} \left[(P_3 + Z_{47})^2 + (P_4 + Q_{37})^2 + (Z_{15} + Z_{62})^2 + (Z_{16} + Z_{25})^2 \right] + \\ & + \frac{1}{2Q_{56}} \left[(P_5 + Q_{67})^2 + (P_6 + Q_{57})^2 + (Z_{13} + Z_{24})^2 + (Z_{14} + Z_{32})^2 \right] + \\ & + \frac{(Z_{12} + Z_{34} + Z_{56})^2}{2(Q_{12} + Q_{34} + Q_{56})} . \end{aligned} \qquad \text{(A.8)}$$

This is the formula needed in the main text.

[7] The signs of Q_{34} and Z_{34} have been flipped relative to (A.1) so that the background charges can be taken positive. Below Q_{37} and Q_{47} will be similarly flipped.

In the considerations relating to black hole entropy we need a couple of refinements. First, it is customary to include the scalar charge, i.e. the momentum P_7 along the R_7 direction. The resulting term in the eigenvalue equation commutes with all the background terms and anticommuetes with all the terms from the U(1) charges. Additionally, the P_7 is small in the scaling limit, of order l_p^0, so cross-terms between the scalar charge and the U(1) charges are negligible. These facts are sufficient to ensure that P_7 can be included in (A.8) by simple addition.

Next, we want to derive the conformal weight of the left-movers. Roughly, this amounts to the substitution $\epsilon \to \Gamma^7\epsilon$, flipping the eigenvalue of the momentum; alternatively one might flip the five-brane numbers by taking the opposite quantum numbers for Γ^{1234}, Γ^{3456}, and Γ^{1256}. Either way there is a problem, because the eigenvalues of the Γ-matrices are not independent. The physical origin of the problem is that the present theory is chiral, with $(4,0)$ supersymmetry. The conformal weight of the left-movers is therefore *not* given by supersymmetry alone. This contrasts with the treatment of the $D1-D5$ system in [5], where the $(4,4)$ supersymmetry was exploited to deduce both the right-moving and the left-moving weights. In the main text we find the left-moving weights using global symmetries.

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